Experimental Verification of Formulae for Natural Seiche Frequencies in a Two-layer Fluid with the Free Surface

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Abstract

Theoretical formulae for the natural frequencies of oscillations of a two-layer fluid due to surface and interface seiches are experimentally tested. It is shown that the formulae based on the strict dispersion relation of the linear theory of the potential motion are applicable in a wide range of the natural frequencies. The approximate formulae based on the long-wave approximation are also applicable but in a narrower frequency range, which is particularly typical for the natural seiches.

1. Introduction

A seiche is a standing wave in an enclosed or partially enclosed body of water (http://en.wikipedia.org/wiki/Seiche). Seiches generate significant horizontal and vertical velocities of water movements. It is unpleasant for ships and is essential for ecology. Both surface and internal seiches exist in water with stable density stratification. There are the explicit theoretical formulae for the wavelengths and frequencies of the different seiche modes in the case of homogeneous initially quiescent fluid filling a rectangular basin with the flat horizontal bottom (Kochin et al., 1963). The linear theory of the potential motion of fluid is used in the theory so the fluid viscosity is not taken into account. The results of an experimental verification of these formulae are contained in Bukreev (2011). The empirical formulae for surface seiches in basins of more complex forms may be found in Rabinovich, (2009).

The explicit formulae are available in Phillips (1967) and Dotsenko and Miklashevskaya (2010) or can also be derived (see below) in the case of a two-layer fluid. Only a two-layer fluid under a horizontal cover was considered in Phillips (1967). The so-cold long wave approximation in the framework of the cited theory was used in Dotsenko & Miklashevskaya, (2010). The theories give relative amplitudes of seiches only. Particular values of the amplitudes depend on the type of disturbance and its intensity. Verification of the theoretical formulae for the frequency oscillations due to surface and internal seiches in a two-layer fluid in controlled laboratory conditions was the main aim of the present work.

Natural seiches are generated by wind and atmospheric pressure variations, seismic activity, tsunamis, landslides, etc. The total property of these disturbances is their relative brevity. As the results, progressive waves are generated at first (Boegman et al., 2005). Seiches are born after the reflection of the progressive waves and are superimposed on the last. As a rule, the same properties of perturbations and waves are typical for laboratory experiments. This is also true for the present work, in which the waves were generated just as it has been done in the experiments of Thorpe (1971); Horn et al. (2001); Boegman (2004); Boegman et al. (2005). Only internal waves were studied in these experiments. Thorpe (1971) received the classical patterns of the shear instability in a two-layer fluid. Progressive internal surges in the form of the moving hydraulic jump were the main object of attenuation in the experiment of Horn et al. (2001); Boegman (2004).

Both surface and internal seiches are specially extracted in the present experiments and theoretical formulae for the wavelengths and frequencies of seiche oscillations are revised on this refined base. That is the main novelty of the present paper.

2. Theoretical Formulae

The wave numbers k_p of the set of the longitudinal seiches in a basin with horizontal bottom are determined by the following formula:

$$k_p = \pi p / L \tag{1}$$

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where $p = 1, 2, 3, ...$ is the number of the normal mode, L is the basin length. The corresponding oscillation frequencies ω_p can be obtained from the dispersion relation $\omega(k)$ of the linear theory, assuming $k = k_p$ in the relation (k is an arbitrary wave number, ω is the corresponding circular frequency). The strict equation for $\omega(k)$ in the problem considered has the form (Ovsyannikov et al., 1985):

$$(kg)^2(1+\gamma\Theta_1\Theta_2)\omega^4 - kg(\Theta_1+\Theta_2)\omega^2 + \Theta_1\Theta_2(1-\gamma) = 0 \tag{2}$$

where g is the acceleration due to gravity,

$$\Theta_1 = \tanh(kh_1), \quad \Theta_2 = \tanh(kh_2), \quad \gamma = \rho_1/\rho_2 \tag{3}$$

h_1 and h_2 are the depths of the upper and lower layers, ρ_1, ρ_2 are the densities of those layers. Equation (1) at each $k = k_p$ has four roots. The sign of the root determines the direction of wave propagation. Only two positive roots have a physical meaning for standing waves, namely:

$$\omega 1_p = \sqrt{\left[\frac{1}{2}b_p + \left(\frac{1}{4}b_p^2 - c_p\right)^{0.5}\right]k_p g} \tag{4}$$

$$\omega 2_p = \sqrt{\left[\frac{1}{2}b_p - \left(\frac{1}{4}b_p^2 - c_p\right)^{0.5}\right]k_p g} \tag{5}$$

where

$$b_p = \frac{\Theta 1_p + \Theta 2_p}{1 + \gamma\Theta 1_p\Theta 2_p}, \quad c_p = \frac{(1+\gamma)\Theta 1_p\Theta 2_p}{1 + \gamma\Theta 1_p\Theta 2_p},$$

Formulae (3) and (4) determine the natural frequencies of the surface and internal oscillations, accordingly.

When hyperbolic tangent in equation (3) can be replaced by its argument (that is the long wave approximation), the following approximate formulae are obtained in Dotsenko and Miklashevskaya, (2010):

$$\omega 1_p \approx \sqrt{\frac{2\varepsilon g h_1 h_2 p^2 \pi^2}{L^2\left(H - \sqrt{H^2 - 4\varepsilon h_1 h_2}\right)}} \tag{6}$$

$$\omega 2_p \approx \sqrt{\frac{2\varepsilon g h_1 h_2 p^2 \pi^2}{L^2\left(H + \sqrt{H^2 - 4\varepsilon h_1 h_2}\right)}} \tag{7}$$

where $H = h_1 + h_2$, $\varepsilon = 1 - \rho_1/\rho_2 = 1 - \gamma$.

If the density stratification is weak ($\varepsilon \ll 1$), formulae (6) and (7) take the next form (Dotsenko & Miklashevskaya, 2010):

$$\omega 1_p \approx \sqrt{\frac{p^2 \pi^2 g H}{L^2}} \qquad (8)$$

$$\omega 2_p \approx \sqrt{\frac{p^2 \pi^2 \varepsilon g h_1 h_2}{L^2 H}} \qquad (9)$$

The mutual influence of the surface and internal seiches is not taken into account in formulae (8) and (9). It should be noted that Coriolis acceleration is taken into consideration in Dotsenko and Miklashevskaya (2010). The corresponding term in formulae (6)–(9) is omitted.

When a transition layer with thickness $\delta << H$ exists between the upper and lower layers, the values $\omega 2_p$ can be estimated by the next approximate formula by Phillips (1967):

$$\omega 2_p \approx \sqrt{\frac{\varepsilon g k_p}{k_p \delta + \mathrm{ctan}\left(k_p h_1\right) + \mathrm{ctan}\left(k_p h_2\right)}} \qquad (10)$$

3. Experiment and Comparison with the Theory

The diagram of experimental setup and fixed rectangular coordinate system are presented in Figure 1. The rectangular basin made of perspex had the length $L = 1.98$ m, the width $B = 0.06$ m and its bottom was flat. The basin could rotate in vertical direction around the support *5* which was located at the distance $l = 1.55$ m from the coordinate system origin (see Figure 1). An analogous setup was used in the experiments of Horn et al. (2001) and Boegman (2004). The basin dimensions were large enough for the similarity of wave processes on the Froude criteria and the self-preservation on the Reynolds criteria to be true. Particularly, this was confirmed by the results of comparison of the received experimental data with the theory in which the Froude criteria is the only similarity criterion (see below).
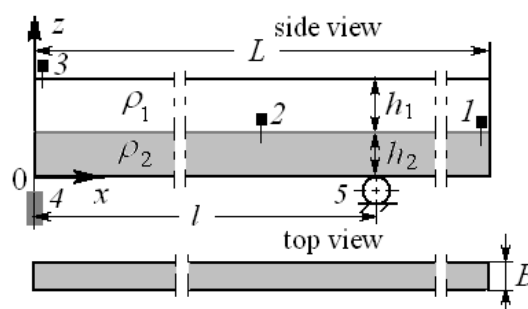
Figure 1. Diagram of the experimental setup

Notes: *1*, *2*, *3*, Wavegauges; *4*, Device for rotate; *5*, Support.

The basin was firstly filled by a layer of tap water of density ρ_1 up to h_1 in depth. After that, a weak glycerin solution with the density $\rho_2 > \rho_1$ was slowly supplied into the basin bottom thus displacing the buoyant fresh water layer. The glycerin solution viscosity exceeded the tap water viscosity no more than 2.5%. The depth of the transition layer δ

after the filling of two layers was about 1 mm. Then the basin was rotated upwards by the angle β with the help of device *4* (see Figure 1). When any movement of water was stopped, the basin was rotated to the horizontal position to generate surface and internal waves. The angle β and the time τ of the basin displacement to the horizontal position are substantial parameters of the problem. Variation of these parameters allows one to obtain different waves (Boegman, 2004). Such combination of the parameter was most suitable for the aim of the present work when waves are smooth and the relative contribution of progressive waves is small. Below, there are two relevant examples. They will be named as Experiment 1 and Experiment 2.

Vertical oscillations ζ of the free surface and interface were recorded by the resistive wavegauges *1*, *2*, and *3* (see Figure 1). The wavegauges could register the oscillations, which exceed 0.2 mm, without substantial errors. Their natural frequency was about 4 Hz. It is shown below that the highest frequencies in the oscillation spectra were less than 1 Hz. The wavegauge *1* was located at the distance of 3 cm from the right butt end of the basin. It registered internal waves. The wavegauge *2* was located in the basin midst and also registered internal waves. The wavegauge *3* was located at the distance of 3 cm from the basin left butt end and registered surface waves.

The wavegauges registered the summary oscillations due to all normal modes. The spectral analysis was performed to determine frequencies of individual modes. The algorithm of Fast Fourier Transformation (FFT) of ζ-oscillations was used for this aim. Analysis was performed with the help of the software application Origin computer product. There are two versions of FFT in Origin named as FFT-amplitude and. FFT-energy. The FFT-amplitude version was more appropriate for unsteady deterministic processes, which are considered in the present paper. The FFT-energy version is used in an analysis of steady random processes, particularly in turbulent fluctuations. The energy of internal waves has been considered by Boegman (2004) and Boegman et al. (2005).

Two problems should be solved at a spectral analysis. One problem is a choice of a time interval for the analysis. Fading ζ-oscillations take place in the experiments and a value of time, above which all ζ <0.2 mm, was used as the upper boundary of the interval. Second problem is the next. Finiteness of the time interval leads to the fact that the product of ζ function and the Heaviside step function (but not the function on demand) is analyzed. So-cold smoothing algorithm is used in order to eliminate the corresponding error. The software application Origin offers a choice of five smoothing algorithms. In the case of ζ-oscillations considered, discrete frequencies in spectra was segregated most clearly by using the "Blackman" algorithm,

Synchronous records of the free surface and interface oscillations in Experiment 1 and Experiment 2 are given in Figures 2(a) and 2(b), respectively. Only given parameters β and τ were different in these experiments. Their values are given in the legend to Figure 2. Other parameters, in particular h_1 =0.06 m, h_2 =0.04 m, ρ_1 =1000 kg/m³, ρ_2 =1009.2 kg/m³, were unchanged. In Figure 2, time t is referred to the start of the basin movement to the horizontal position. The functions $\zeta_i(t)$ (i =1, 2, 3) are the oscillations which were registered by the wavegauges *1*, *2,* and *3,* respectively. Over the time, these functions attain constant asymptotic values corresponding to the state of rest. These asymptotic values are equal to zeros in Figure 2.
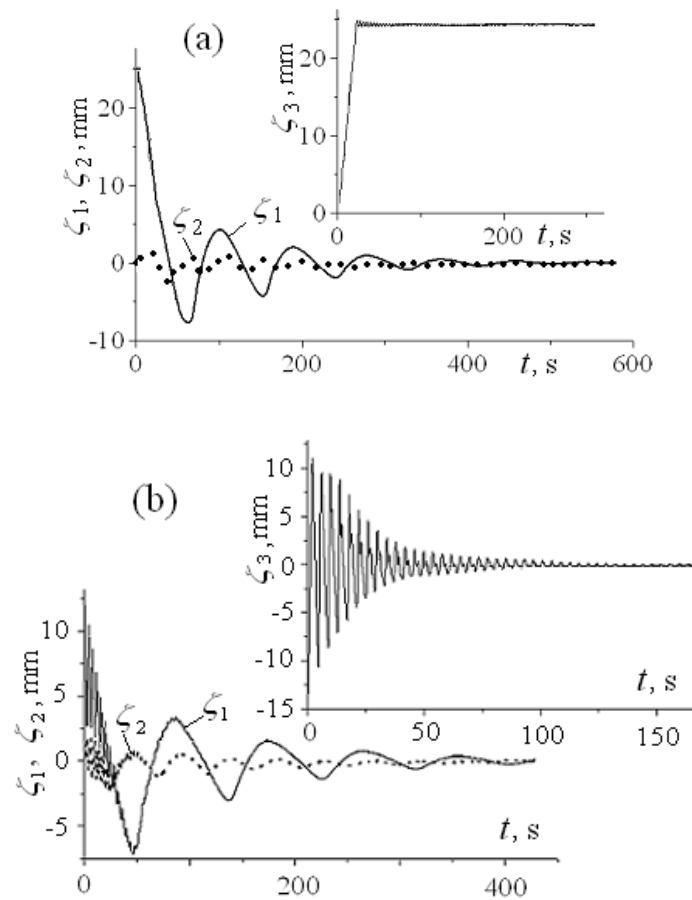
Figure 2. Oscillations of the interface (ζ_1, ζ_2) and surface (ζ_3)

Notes: (a) Experiment 1: β =1.22^o, τ =25 s; (b) Experiment 2: β =0.42^o, τ =2 s.

In Experiment 1, the free surface attained monotonically its asymptotical level almost just after the rotation of the basin to the horizontal position and the surface waves were very small. In Experiment 2, the surface waves were significant and affected the internal waves. All seiche modes have the antinodes at the basin ends. As the result, the summary oscillations are maximal here. In the basin middle, all odd seiche modes have the nodes so the oscillations are minimal here. The viscosity of water is not considered in the theory. Data given in Figure 2 showed that the waves were degenerated and the surface waves were degenerated faster than the internal ones.

The influence of the surface waves on the internal ones is illustrated by two images shown in Figure 3. The images are received in Experiment 1 with the values of τ =0.3 s at two t values. The bottom layer is colored with aquatic solution of commercial ink (3 milliliters of commercial ink in 1 liter of water). Solid particles are visible in the images. That is wax (paraffin) weighted by aluminum powder. Some particles are lighter then fresh water and track the movement of the free surface. Other particles have the density near the density of salt water in the lower layer. They tracked the movement in the neighborhood of the

interface. It should be noted that the pointed way of preparation of relatively large marked particles was the most successful among a number of others ways. In the case of stable stratified salt water, such particles tracked the corresponding isopycnic surface for a few days. The distance between the vertical solid lines in Figure 3 is equal to 0.2 m. The dash vertical line in Figure 3 makes the position of the node of second normal mode seiche.
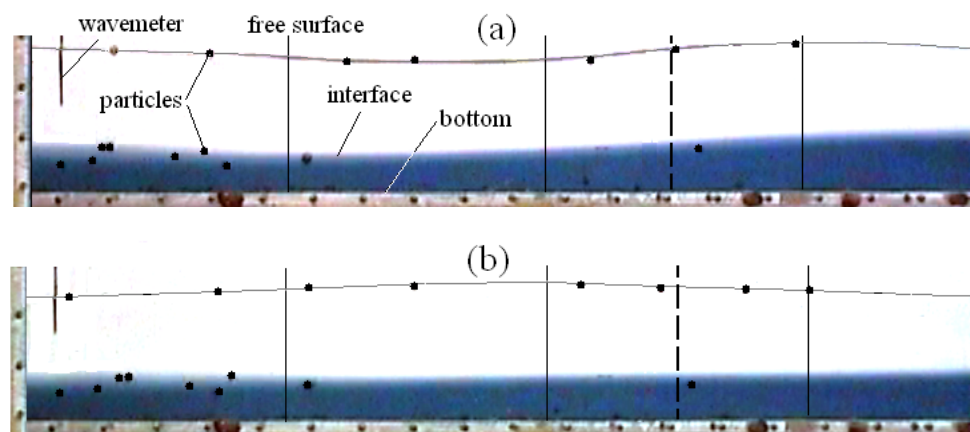
Figure 3. Example of influence of surface wave on internal wave at $\beta = 1.22^{o}$ and $\tau = 0.3$ s

Notes: (a) $t = 10.3$ s; (b) $t = 12.3$ s.

The dimensionless modules $\left| s_i^0(f) \right|$ of the spectra $S_i(f)$ of the functions $\zeta_i(t)$ are presented in Figures 4(a) and 4(b) (f is a frequency). The maximum of the dimension experimental spectrum at $i = 1$ and $j = 1$ is used as the scale for all dimensionless spectra. Numerical values of these scales are different in Figure 4(a) and Figure 4(b). The modulus $\left| S_3^0(f) \right|$ is not shown in Figure 4(a) whereas the surface waves in Experiment 1 were too weak (see ζ_3 in Figure 2(a)). The spectra have a number of the maxima at the frequencies f_j^e ($j = 1, 2, 3, \ldots$). These frequencies are compared below with the theoretical frequencies of the seiche oscillations. There is a specific maximum A in Figure 4. It is due to the unsteady nature of the waves and it is not described by the theory. The functions $\left| S_2^0(f) \right|$ have notable maxima only at the even values of j since the corresponding wavegauge was located in the theoretical nodal point for all odd seiche modes.
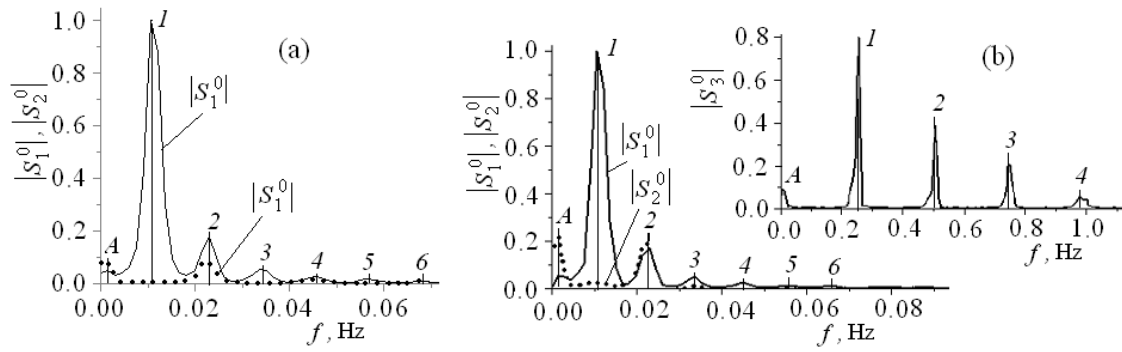
Figure 4. Modules of the spectra of the oscillations ζ_i shown in Figure 2

Notes: (a) Experiment 1: (b) Experiment 2.

The results of analysis of the correlation between the experimental frequencies f_j^e and the theoretical frequencies $f_p = \omega_p / 2\pi$ are presented in Figure 5. The following new symbols are used in Figure 5: f_p^{in} is the frequency of the theoretical internal seiche (Figure 5(a)) and f_p^{sur} is the frequency of the theoretical surface seiche (Figure 5(b)). The condition $p = j$ is satisfied for the discrete points in Figure 5. Formula (8) gives almost the same results as formula (6) in the considered example. This is also true with respect to formulae (7) and (5).

If the correlation is ideal, the discrete points to be grouped around the coordinate angle bisector which is shown by the straight line in Figure 4. The data presented in Figure 4 show that the correlation is practically ideal at least up to $p=10$ when the strict formulae (3) and (4) are used. The approximate formulae (5) and (6) give acceptable results up to $p=5$ only. This is natural since they are based on the long wave approximation. Approximate formula (9) is acceptable in a wider range of the interfacial seiche natural frequencies.
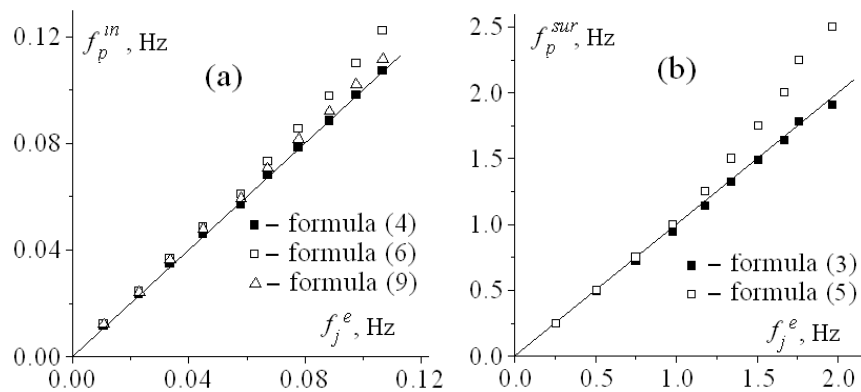
Figure 5. Correlation of the experimental (abscissa) and theoretical (ordinate) frequencies

Notes: (a) Internal wave ζ_1 in Experiment 1; (b) Surface wave ζ_3 in Experiment 2.

One result of the wave degradation analysis is presented in Figure 6. The interfacial oscillations $\zeta_1(t)$ near the basin right butt end in Example 1 was used in the analysis. The upper $Z_u(t)$ and lower $Z_l(t)$ envelopes and the oscillations $\zeta_1(t)$ are shown in Figure 6. The envelopes were described well enough by the following function:

$$Z = E \exp(-t/t_0) \tag{11}$$

where E and t_0 are parameters of the $Z(t)$ function. They depend on the given parameters of the problem. The value $\alpha = 1/t_0$ is called the attenuation coefficient in the oscillation theory. In the example considered, $E = 11.1$ mm, $t_0 = 118$ s ($\alpha = 0.00847$ s^{-1}) for the upper envelope and $E = -13.2$ mm, $t_0 = 130$ s ($\alpha = 0.00769$ s^{-1}) for the lower envelope.
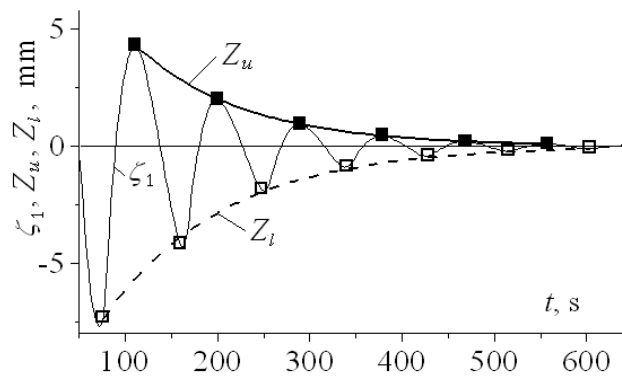
Figure 6. Envelopes of the internal wave ζ_1 in Experiment 1

The surface seiches attenuated faster than the internal ones. For example, the parameters of the upper envelope of the oscillations $\zeta_3(t)$ in Example 2 (see Figure 2(b)) were the following: $E = 13.9$ mm, $t_0 = 26$ s, i.e., the surface seiche attenuated approximately 4.5 times faster than the internal one.

4. Discussion

Natural water bodies have very complex geometry so any simplification of mathematical models of seiches is desirable. The long-wave approximation is used in all examples of natural seiche numerical computations (Dotsenko & Miklashevskaya, 2010; Terletska et al., 2011). The results of the present study confirm the validity of this approximation. As the data given in Figure 4 show, this approximation is good for internal waves up to the frequency $\omega = 2\pi f \leq 0.3$ (rad/s) at last. In the accordance with dispersion relation (1), the corresponding range of wave numbers is $k \leq 6.4$ (m^{-1}) and the corresponding range of wavelengths is $(\lambda = 2\pi/k) \geq 1$ (m). The same value is also approximately true for the surface waves. Natural seiches are much longer than 1 m.

The good coincidence of the experimental and strict theoretical frequencies in Figure 4 justifies the most significant simplification of the theory, namely: using the potential motion

model. From this result, it follows that the fluid viscosity did not affect the length and frequency of the seiche even in the laboratory experiment. The theoretical and experimental study of the water viscosity influence on the dispersion relation for a two-layer fluid was performed in the work of Borodina et al. (1982). It has been shown that this influence is noticeably manifested only at the wavelengths less than 1.5 cm.

5. Conclusions

The validity of the linear theory of potential motion at computations of the wavelengths and oscillation frequencies of the surface and internal seiches is confirmed by the example of a two-layer fluid. Relevancy of the long-wave approximation in the mentioned theory for natural seiches is also confirmed. Calculations of the seiche amplitudes, in particular, their attenuation, should be performed on the base of a mathematical model that takes into account specific disturbances and dissipation of energy.

Acknowledgement

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Symbols

A Designation of one of maxima in Figure 3

B Channel width (m)

b_p Dimensionless parameter

c_p Dimensionless parameter

E Parameter (mm)

f Frequency (Hz)

f_p Theoretical normal frequency (Hz)

f_p^{in} Theoretical normal frequency of internal mode (Hz)

f_p^{sur} Theoretical normal frequency of surface mode (Hz)

f_j^e Frequency of experimental maximum (Hz)

g Acceleration due to gravity (m/s^2)

H Initial overall depth of fluid (m)

h_1, h_2 Initial depths of the upper and lower fluids respectively (m)

$i = 1, 2, 3$ Index

$j = 1, 2, 3, \ldots$ Index

k Wave number ($2\pi \cdot \mathrm{m}^{-1}$)

k_p Normal wave number ($2\pi \cdot \mathrm{m}^{-1}$)

L Basin length (m)

l Longitudinal coordinate of support (m)

$p = 1, 2, 3, \ldots$ Index

S_i Spectrum ($\mathrm{mm} \cdot \mathrm{s}$)

$\left|S_1^0\right|$, $\left|S_2^0\right|$, $\left|S_3^0\right|$ Modules of individual dimensionless spectra

t Time (s)

t_0 Characteristic attenuation time (s)

x Longitudinal coordinate (m)

z Vertical coordinate (m)

α Attenuation coefficient (s^{-1})

β Angle (degree)

δ Characteristic thickness of intermediate layer (mm)

ε Dimensionless parameter

γ Dimensionless parameter

λ Wavelength (m)

ρ_1, ρ_2 Densities (kg/m^3)

τ Characteristic time of rotation (s)

ω Circular frequency (rad/s)

ω_p Normal circular frequency (rad/s)

$\omega 1_p$ Normal circular frequency of surface mode (rad/s z)

$\omega 2_p$ Normal circular frequency of internal mode (rad/s)

ζ Vertical oscillation (mm)

ζ_i Vertical oscillations registered by wavegauges (mm)

ζ_1, ζ_2, ζ_3 Individual oscillations (mm)

Θ_1, Θ_2 Dimensionless parameters

Z Envelope (mm)

Z_u, Z_l Upper and lower envelopes (mm)

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